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Totally positive functions through nonstationary subdivision schemes

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Abstract

In this paper a new class of nonstationary subdivision schemes is proposed to construct functions having all the main properties of B-splines, namely compact support, central symmetry and total positivity. We show that the constructed nonstationary subdivision schemes are asympotically equivalent to the stationary subdivision scheme associated with a B-spline of suitable degree, but the resulting limit function has smaller support than the B-spline although keeping its regularity.
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1. Introduction

The power of *subdivision schemes* has been extensively established in several contexts, like, just to mention two well-known examples, the design of smooth curves and surfaces and the generation of refinable functions and wavelets. Subdivision schemes basically are iterative schemes based on simple refinement rules generating denser and denser sequence of points convergent to a continuous curve or surface. A celebrated example is given by B-spline stationary subdivision schemes that can be used to generate spline curves.

An important review of the different subdivision schemes, which range from stationary (i.e., the refinement rules do not depend on the recursion level) to nonstationary, from uniform (i.e., the refinement rules do not vary from point to point) to nonuniform, from binary (i.e., the number of points is 'doubled' at each iteration) to any a-rity, can be found in [3].

In the variety of subdivision algorithms investigated nowadays, our interest is in the direction of nonstationary schemes whose importance is shown, just to give an example, in the construction of the up-function. Actually, the up-function having support $[0, 2]$ and regularity C^∞ is the limit of a nonstationary subdivision scheme based on stationary B-spline masks (see, for example, [1]).

The goal of this paper is to use nonstationary subdivision schemes for constructing symmetric and totally positive functions having the same support but higher smoothness than B-splines. In particular, we are going to construct totally positive symmetric functions with support of length n and smoothness C^{n-1}.

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The rest of the paper is organized as follows. In Section 2 we list all the basic facts about nonstationary subdivision schemes needed in the paper. In Section 3 we introduce the new class of totally positive functions generated via nonstationary subdivision schemes and we discuss some of their properties. Finally, in Section 4, some graphs of the constructed functions together with the curves generated by the corresponding nonstationary subdivision schemes are displayed.

2. Preliminaries

Any *nonstationary subdivision scheme* can be represented by an infinite sequence of refinement masks $\{\boldsymbol{a}^k\}_{k\geqslant 0}$, $k\in\mathbb{N}^+$. We assume that any sequence $\boldsymbol{a}^k=\{a^k_\alpha\}_{\alpha\in\mathbb{Z}}$ is of real numbers and have finite support for all $k\geqslant 0$. The *k-level subdivision operator* associated with the *k-level mask* $\boldsymbol{a}^k$ is

$$S_{\boldsymbol{a}^k}:\ell(\mathbb{Z})\to\ell(\mathbb{Z}),$$
$$(S_{\boldsymbol{a}^k}\,\boldsymbol{\lambda})_\alpha:=\sum_{\beta\in\mathbb{Z}}a^k_{\alpha-2\beta}\,\lambda_\beta,\quad \alpha\in\mathbb{Z}, \tag{2.1}$$

where $\boldsymbol{\lambda}=\{\lambda_\alpha\}_{\alpha\in\mathbb{Z}}\in\ell(\mathbb{Z})$, the linear space of real sequences indexed by $\mathbb{Z}$. The nonstationary subdivision scheme consists of the subsequent application of $S_{\boldsymbol{a}^0},\ldots,S_{\boldsymbol{a}^k}$ generating the scalar sequences

$$\boldsymbol{\lambda}^0:=\boldsymbol{\lambda},\quad \boldsymbol{\lambda}^{k+1}:=S_{\boldsymbol{a}^k}\,\boldsymbol{\lambda}^k\ \text{ for } k\geqslant 0. \tag{2.2}$$

If $\{\boldsymbol{a}^k=\boldsymbol{a}\}_{k\geqslant 0}$ the scheme is termed *stationary*.

Obviously, the process could also be started from any fixed level m, that is by subsequent application of $S_{\boldsymbol{a}^m},\ldots,S_{\boldsymbol{a}^{m+k}}$ for $k\geqslant 0,\ m\geqslant 0$.

In order to investigate the subdivision properties, it is convenient to consider the Laurent polynomials associated to the refinement masks $\boldsymbol{a}^k$, i.e.,

$$a^k(z)=\sum_{\alpha\in\mathbb{Z}}a^k_\alpha z^\alpha. \tag{2.3}$$

A subdivision scheme is L_∞-*convergent* if, for any $\boldsymbol{\lambda}$ *in* $\ell^\infty(\mathbb{Z})$, the linear space of bounded scalar sequences indexed by $\mathbb{Z}$, there exists a continuous vector-valued function $f_{\boldsymbol{\lambda}}$ (depending on the starting sequence $\boldsymbol{\lambda}$) satisfying

$$\lim_{k\to\infty}\left\|f_{\boldsymbol{\lambda}}\left(\frac{\cdot}{2^k}\right)-\boldsymbol{\lambda}^k\right\|_\infty=0 \tag{2.4}$$

and $f_{\boldsymbol{\lambda}}\neq 0$ for at least some initial data $\boldsymbol{\lambda}$. Here, the symbol $f_{\boldsymbol{\lambda}}(\cdot/2^k)$ abbreviates the scalar sequence $\{f_{\boldsymbol{\lambda}}(\alpha/2^k)\}_{\alpha\in\mathbb{Z}}$ and $\|\cdot\|_\infty:=\sup_{\alpha\in\mathbb{Z}}|\lambda_\alpha|$.

Two nonstationary subdivision schemes with masks $\{\boldsymbol{a}^k\}_{k\geqslant 0}$ and $\{\boldsymbol{b}^k\}_{k\geqslant 0}$ are said to be *asymptotically equivalent*, in symbols $\{\boldsymbol{a}^k\}_{k\geqslant 0}\approx\{\boldsymbol{b}^k\}_{k\geqslant 0}$, if for some fixed $L\in\mathbb{Z}$ it results

$$\sum_{k=\max\{0,-L\}}^{\infty}\|\boldsymbol{a}^{k+L}-\boldsymbol{b}^k\|_\infty<\infty, \tag{2.5}$$

where $\|\boldsymbol{a}^k-\boldsymbol{b}^j\|_\infty=\max_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}}|a^k_{\alpha-2\beta}-b^j_{\alpha-2\beta}|$.

This concept allows one to derive convergence properties of a given nonstationary subdivision scheme from those of an asymptotic equivalent subdivision scheme known to be convergent. To this respect, some results in [3] can be concisely stated as follows.

Theorem A. *Let* $\{\boldsymbol{a}^k\}_{k\geqslant 0}\approx\{\boldsymbol{b}^k:=\boldsymbol{b}\}_{k\geqslant 0}$ *be two asymptotically equivalent subdivision schemes with the latter being a stationary subdivision. If the stationary subdivision* $\{\boldsymbol{b}\}$ *is convergent, then* $\{\boldsymbol{a}^k\}_{k\geqslant 0}$ *is convergent as well.*

With the help of Laurent polynomial representation in [9] Levin gave sufficient conditions for convergence and smoothness investigation of nonstationary univariate subdivision schemes. These conditions are based on contractivity of the ℓ-iterate of the subdivision operator. For completeness, we shortly recall its result.

Theorem B. *Let* $\{a^k(z)\}_{k\geqslant 0}$ *the Laurent polynomials associated with the masks* $\{\boldsymbol{a}^k\}_{k\geqslant 0}$. *Let* $\{b^{k,r}(z),\ 1\leqslant r\leqslant N+1\}$ *be the Laurent polynomials recursively defined as*

$$b^{k,r+1}(z)=\frac{2^{r+1}b^{k,r}(z)}{z+1},\quad 0\leqslant r\leqslant N, \tag{2.6}$$

where

$$b^{k,0}(z)=a^k(z). \tag{2.7}$$

If the scheme having Laurent polynomials $\{b^{k,N+1}(z)\}_{k\geqslant 0}$ *is convergent, the scheme with masks* $\{\boldsymbol{a}^k\}_{k\geqslant 0}$ *is* C^N.

For any convergent nonstationary subdivision scheme with masks $\{\boldsymbol{a}^k\}_{k\geqslant 0}$, one gets a family of *basic limit functions* each defined by

$$\phi_m:=\lim_{k\to\infty} S_{\boldsymbol{a}^{m+k}}S_{\boldsymbol{a}^{m+k-1}}\cdots S_{\boldsymbol{a}^m}\,\boldsymbol{\delta}_0,\quad m\geqslant 0, \tag{2.8}$$

where $\boldsymbol{\delta}_0$ is the delta-sequence, i.e., $\boldsymbol{\delta}_0=\{\delta_{\alpha,0}\}_{\alpha\in\mathbb{Z}}$. The functions ϕ_m, $m\geqslant 0$, are solutions to the functional equations

$$\phi_m(x)=\sum_{\alpha\in\mathbb{Z}} a^m_\alpha\,\phi_{m+1}(2x-\alpha),\quad x\in\mathbb{R},\quad m\geqslant 0, \tag{2.9}$$

which, following [3,4], we denote by *nonstationary refinement equations*. In the stationary case Eq. (2.9) reduce to the *refinement equation*

$$\phi(x)=\sum_{\alpha\in\mathbb{Z}} a_\alpha\phi(2x-\alpha),\quad x\in\mathbb{R}. \tag{2.10}$$

The nonstationary refinement equation (2.9) are related to the *nonstationary cascade algorithm* defined as follows [4].

Choose a starting sequence of continuous functions $\{h_{m,0}\}_{m\geqslant 0}$ in $L^2(\mathbb{R})$ such that, for any m, supp $h_{m,0}\subseteq K\subset\mathbb{R}$ with $\bigcup_{\alpha\in\operatorname{supp}\boldsymbol{a}^k}\frac{1}{2}(K+\alpha)\subset K$. Furthermore, assume that there exists a function $\widetilde{h}_0$ in $L^2(\mathbb{R})$ such that

$$h_{m,0}\to\widetilde{h}_0 \text{ as } m\to\infty \quad\text{and}\quad \widehat{h}_{m+k,0}\left(\frac{u}{2^k}\right)\to 1 \text{ as } k\to\infty, \tag{2.11}$$

uniformly in m and locally uniformly in u; these conditions are easily satisfied if the functions $h_{m,0}$ are independent of m and $\widehat{h}_{m,0}(0)=1$ for any $m\geqslant 0$.

The nonstationary cascade algorithm associated with the masks $\{\boldsymbol{a}^k\}_{k\geqslant 0}$ generates the sequences of functions $\{h_{m,k}\}_{m\geqslant 0}$, $k>0$, by the following equations:

$$h_{m,k}(x)=\sum_{\alpha\in\mathbb{Z}} a^m_\alpha\,h_{m+1,k-1}(2x-\alpha),\quad x\in\mathbb{R},\ \ m\geqslant 0,\ \ k>0. \tag{2.12}$$

The link with the nonstationary subdivision scheme with masks $\{\boldsymbol{a}^k\}_{k\geqslant m}$, $m\geqslant 0$, is expressed by the relations

$$h_{m,k}(x)=\sum_{\alpha\in\mathbb{Z}} s^{m,k}_\alpha\,h_{m+k,0}(2^kx-\alpha),\quad x\in\mathbb{R},\ \ m\geqslant 0,\ \ k>0, \tag{2.13}$$

where the sequences $\mathbf{s}^{m,k}=\{s^{m,k}_\alpha\}_{\alpha\in\mathbb{Z}}$, $m\geqslant 0$, $k>0$, are defined as

$$\mathbf{s}^{m,k}:=S_{\boldsymbol{a}^{m+k-1}}\cdots S_{\boldsymbol{a}^m}\,\boldsymbol{\delta}_0. \tag{2.14}$$

Assuming $\{\boldsymbol{a}^k\}_{k\geqslant 0}\approx\boldsymbol{a}$, where $\boldsymbol{a}$ is the mask associated with a stationary subdivision scheme, and $\{h_{m,0}\}_{m\geqslant 0}$ a stable sequence, in [4] it is shown that

$$\phi_m=\lim_{k\to\infty} S_{\boldsymbol{a}^{m+k}}\cdots S_{\boldsymbol{a}^m}\,\boldsymbol{\delta}_0 \;\Leftrightarrow\; \lim_{k\to\infty}\|h_{m,k}-\phi_m\|_\infty=0,\quad m\geqslant 0. \tag{2.15}$$

3. A new class of functions generated via nonstationary schemes

We start from the family of GP stationary subdivision algorithms, namely subdivision schemes to be used to construct the so called GP functions, introduced by the two last authors in [6]. The class of GP functions is made of compactly supported, centrally symmetric and totally positive refinable functions with prescribed smoothness.

We recall that a function F is said to be *totally positive* [7] if there results

$$\det_{l,j=1,\dots,p} F(x_l - \alpha_j) \geqslant 0 \tag{3.1}$$

for all $x_1 < \cdots < x_p$ and integers $\alpha_1 < \cdots < \alpha_p$, $p \geqslant 1$. The importance of totally positive functions in CAGD applications is related to the *variation diminishing* [7] property they enjoy; as a consequence, any curve $C(t) = \sum_{\alpha\in\mathbb{Z}} P_\alpha F(t-\alpha)$, constructed on a sequence of control points $\{P_\alpha \in \mathbb{R}^2\}_{\alpha\in\mathbb{Z}}$, has *shape preserving properties*.

A stationary GP subdivision algorithm is given by the compactly supported subdivision masks $\boldsymbol{a}^{(n,h)} = \{a_\alpha^{(n,h)}\}_{\alpha=0}^{n+1}$ defined as

$$a_\alpha^{(n,h)} = \frac{1}{2^h}\left[\binom{n+1}{\alpha} + 4(2^{h-n}-1)\binom{n-1}{\alpha-1}\right], \quad \alpha = 0,\dots,n+1, \tag{3.2}$$

where $h \geqslant n \geqslant 2$ is a real parameter. Here, and in the rest of the paper,

$$\binom{n}{\alpha} = 0 \quad \text{for } \alpha < 0 \text{ or } \alpha > n.$$

The corresponding symbol, namely $a^{(n,h)}(z) := \sum_{\alpha\in\mathbb{Z}} a_\alpha^{(n,h)} z^\alpha$, is

$$a^{(n,h)}(z) = \frac{1}{2^h}(z+1)^{n-1}(z^2 + 2(2^{h-n+1}-1)z + 1). \tag{3.3}$$

Due to the factor $(1/2^{n-1})(z+1)^{n-1}$, the stationary subdivision scheme associated with any mask $\boldsymbol{a}^{(n,h)}$ generates C^{n-2} functions. It also follows that the refinable function $\varphi^{(n,h)}$ generated when starting the subdivision process with the delta-sequence, is compactly supported, totally positive with smoothness C^{n-2}. Moreover, whenever $n = h$, $\varphi^{(n,h)}$ reduces to the B-splines of degree n since $a^{(n,h)}(z)$ reduces to $(1/2^n)(z+1)^{n+1}$. Another important special case is the limit case we get when h goes to infinity. In fact, the limit symbol is

$$\begin{aligned} a^{(n,\infty)}(z) &:= \lim_{h\to\infty} \frac{1}{2^h}(z+1)^{n-1}(z^2 + 2(2^{h-n+1}-1)z + 1) \\ &= \frac{1}{2^{n-2}}(z+1)^{n-1} z, \end{aligned} \tag{3.4}$$

which is the symbol of the shifted version of the subdivision mask associated with a B-spline of degree $n-2$. As a consequence, as the parameter h varies from n to ∞, GP functions describe a class of C^{n-2} functions which, although are not splines, range from n-degree B-splines to $n-2$-degree B-splines.

To give an idea of the behavior of the GP functions, in Fig. 1 the graphs of $\varphi^{(3,h)}$ for $h = 3, 4, 8$ are shown. The pictures are obtained by performing five steps of the subdivision algorithm associated with the masks

$$\boldsymbol{a}^{(3,3)} = \frac{1}{2^3}\{1, 4, 6, 4, 1\}, \quad \boldsymbol{a}^{(3,4)} = \frac{1}{2^4}\{1, 8, 14, 8, 1\},$$

$$\boldsymbol{a}^{(3,8)} = \frac{1}{2^8}\{1, 128, 254, 128, 1\}.$$

In order to improve the smoothness of the GP functions here we propose to use a nonstationary subdivision scheme to generate a new family of functions. Practically, we set the additional free parameter h in the mask $\boldsymbol{a}^{(n,h)}$ to have the special form $h := n + (1/k^\mu)$, $\mu > 1$, and we construct an infinite sequence of refinement masks. It means that, for any fixed n, we consider the set of masks $\boldsymbol{a}^k = \{a_\alpha^k\}_{\alpha=0}^{n+1}$, defined for $k = 0$ as

$$a_\alpha^0 = \frac{1}{2^{n-2}}\binom{n-1}{\alpha-1}, \quad \alpha = 1,\dots,n, \tag{3.5}$$

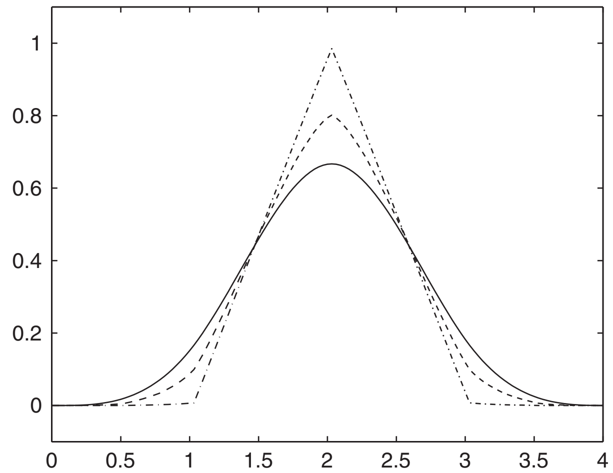

Fig. 1. Graphs of $\varphi^{(3,3)}$ (cubic B-spline) (–), $\varphi^{(3,4)}$(– –) and $\varphi^{(3,8)}$(·–).

and for $k > 0$ as

$$a_\alpha^k = \frac{1}{2^{n+(1/k^\mu)}}\left[\binom{n+1}{\alpha} + 4(2^{1/k^\mu}-1)\binom{n-1}{\alpha-1}\right], \quad \alpha = 0, \ldots, n+1. \tag{3.6}$$

Consequently, the set of symbols we deal with is

$$a^0(z) = \frac{1}{2^{n-2}}(z+1)^{n-1} z \quad \text{for } k=0,$$

$$a^k(z) = \frac{1}{2^{n+(1/k^\mu)}}(z+1)^{n-1}(z^2 + 2(2^{(1/k^\mu)+1}-1)z+1) \quad \text{for } k>0. \tag{3.7}$$

Taking into consideration that the limit mask $\boldsymbol{a}^\infty := \lim_{k\to\infty} \boldsymbol{a}^k$ corresponds to the one of the n-degree B-spline of support $[0, n+1]$, we expect to generate limit functions having shape given by the starting mask and smoothness by the limit one, that is C^{n-1}. This result is better formulated in Theorems 3.2 and 3.3 below where, in order to obtain the smallest support for the basic limit function ϕ_0, we will choose as starting mask $\boldsymbol{a}^0$, i.e., the one associated with the $n-2$-degree B-spline of support $[1, n]$.

Before proceeding we need an auxiliary result.

Proposition 3.1. *For $\mu > 1$ the series $\sum_{k=1}^\infty (1-2^{-1/k^\mu})$ is convergent.*

Proof. For $x > 0$ consider the real function $f(x) = 1-2^{-x}$ which is positive, concave and monotone increasing. By the concavity we have $0 \leqslant f(x) \leqslant f(0) + f'(0)\cdot x$. Equivalently, $0 \leqslant 1-2^{-x} \leqslant x\log(2)$. Form above we get

$$0 \leqslant \sum_{k=1}^\infty (1-2^{-1/k^\mu}) \leqslant \log(2)\sum_{k=1}^\infty \frac{1}{k^\mu},$$

thus, concluding the proof. □

Theorem 3.2. *For any $\mu > 1$, the nonstationary subdivision scheme* (3.5) *and* (3.6) *generates C^{n-1} functions.*

Proof. We start by considering the convergence of the scheme we generate with the masks having symbols

$$b^0(z) = 2z, \tag{3.8}$$

and

$$b^k(z) = \frac{1}{2^{1+(1/k^\mu)}}(z^2 + 2(2^{(1/k^\mu)+1} - 1)z + 1), \quad k > 0. \tag{3.9}$$

Since $b^\infty(z) := \lim_{k\to\infty} b^k(z) = \frac{1}{2}(z+1)^2$, we are going to show that the nonstationary subdivision scheme associated with the masks (3.8) and (3.9) is asymptotically equivalent to the linear B-spline, which implies, by Theorems A and B, that the scheme is convergent and the limit function is C^0. In fact, for $k > 0$

$$b^\infty(z) - b^k(z) = \tfrac{1}{2}(z+1)^2\,(1 - 2^{-1/k^\mu}),$$

hence, from Proposition 3.1, we get $\sum_{k=0}^\infty \|\boldsymbol{b}^k - \boldsymbol{b}^\infty\|_\infty < \infty$, which is the asymptotic equivalence of the nonstationary subdivision scheme $\{\boldsymbol{b}^k\}_{k\geqslant 0}$ and the stationary subdivision scheme associated with linear B-spline. Since $a^k(z) = ((z+1)^{n-1}/2^{n-1})\, b^k(z)$, $k \geqslant 0$, Theorem B allows us to prove the smoothness result. $\square$

Theorem 3.3. *Let* $\phi_m := \lim_{k\to\infty} S_{\boldsymbol{a}^{m+k}} \cdots S_{\boldsymbol{a}^m}\, \boldsymbol{\delta}_0$, $m \geqslant 0$, *be the basic limit functions generated by the nonstationary subdivision scheme* (3.5) *and* (3.6). *The functions* ϕ_m, $m \geqslant 0$, *are centrally symmetric, compactly supported, and totally positive.*

Proof. The proof adapts the idea discussed in [5] to the nonstationary case. To each mask in (3.5) and (3.6) we associate the k-level linear operator $T_{\boldsymbol{a}^k}$ acting on a continuous function F as

$$T_{\boldsymbol{a}^k}\, F(x) := \sum_{\alpha\in\mathbb{Z}} a^k_\alpha\, F(2x - \alpha), \quad x \in \mathbb{R}, \ \ k \geqslant 0. \tag{3.10}$$

For a given starting sequence $\{h_{m,0}\}_{m\geqslant 0}$, the repeated application of $T_{\boldsymbol{a}^k}$ gives

$$h_{m,k} = T_{\boldsymbol{a}^m} \circ \cdots \circ T_{\boldsymbol{a}^{m+k}}\, h_{m,0}, \quad m \geqslant 0, \ \ k > 0,$$

so that, by definition,

$$h_{m+1,k-1} = T_{\boldsymbol{a}^{m+1}} \circ \cdots \circ T_{\boldsymbol{a}^{m+k}}\, h_{m+1,0}, \quad m \geqslant 0, \ \ k > 0.$$

If $h_{m,0}$ is independent of m, it follows $h_{m,k} = T_{\boldsymbol{a}^m} h_{m+1,k-1}$ or, explicitly,

$$h_{m,k}(x) = \sum_{\alpha\in\mathbb{Z}} a^m_\alpha\, h_{m+1,k-1}(2x - \alpha), \quad x \in \mathbb{R}, \ \ m \geqslant 0, \ \ k > 0. \tag{3.11}$$

Now, let us assume $h_{m,0} = B_1$, $m \geqslant 0$, where B_1 is the B-spline of degree 1. Since the starting sequence $\{h_{m,0}\}$ is stable and satisfies (2.11), the cascade algorithm (3.11) is convergent.

Due to the central symmetry of each mask in the sequence $\{\boldsymbol{a}^k\}_{k\geqslant 0}$ and of B_1, from above we get central symmetry of $h_{m,k+1}$ for each $m \geqslant 0$ and each $k > 0$ with respect to the point $n + 1/2$. Thus, the same property holds for the limit functions $\phi_m = \lim_{k\to\infty} h_{m,k}$, too.

As regards to the support, we can use the results given in [3, p. 6]. In fact, assuming $[l(k), r(k)]$, $k \geqslant 0$, are the supports of the k-level masks associated with a nonstationary subdivision scheme, the support of the limit basic function ϕ_m is proved to be included in

$$[L_m, R_m] := \left[\sum_{k=m}^{\infty} 2^{m-k-1} l(k),\ \sum_{k=m}^{\infty} 2^{m-k-1} r(k)\right].$$

In our case $l(0) = 1$ and $r(0) = n$ while $l(k) = 0$ and $r(k) = n + 1$, for $k > 0$. Thus, in case $m = 0$ the left endpoint is $L_0 = \frac{1}{2}$ while the right endpoint is

$$R_0 = \sum_{k=0}^{\infty} 2^{-k-1} r(k) = \frac{n}{2} + \frac{n+1}{2} \underbrace{\sum_{k=1}^{\infty} 2^{-k}}_{1} = n + \frac{1}{2},$$

for $m > 0$, $L_m = 0$ while the right endpoint is

$$R_m = \sum_{k=m}^{\infty} 2^{m-k-1} r(k) = (n+1)2^{m-1} \underbrace{\sum_{k=m}^{\infty} 2^{-k}}_{\frac{1}{2^{m-1}}} = n+1.$$

Finally, we prove that the functions $\phi_m,\ m \geqslant 0$, are totally positive. In fact, from (3.11) and using the Cauchy–Binet formula we get

$$\begin{aligned} H_{m,k+1}\begin{pmatrix} x_1, \ldots, x_p \\ \alpha_1, \ldots, \alpha_p \end{pmatrix} &:= \det_{l,r=1,\ldots,p} h_{m,k+1}(x_l - \alpha_r) \\ &= \sum_{\beta_1 < \cdots < \beta_p} A_m \begin{pmatrix} 2\alpha_1, \ldots, 2\alpha_s \\ \beta_1, \ldots, \beta_s \end{pmatrix} H_{m+1,k} \begin{pmatrix} 2x_1, \ldots, 2x_s \\ \beta_1, \ldots, \beta_s \end{pmatrix}, \end{aligned}$$

where

$$A_k \begin{pmatrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_p \end{pmatrix} := \det_{r,l=1,\ldots,p} a^k_{\beta_r - \alpha_l}, \tag{3.12}$$

with $\alpha_1 < \cdots < \alpha_p$ and $\beta_1 < \cdots < \beta_p$. Since the symbol associated to any mask $\boldsymbol{a}^k$ is a left-half plane stable polynomial, for any $p > 0$ and for any sequences $\{\alpha_k\}$, $\{\beta_k\}$, the determinants in (3.12) are nonnegative [8]. Moreover, $\{h_{m,0}\}$ are totally positive functions, then, by induction, any $h_{m,k}$, $k > 0$, $m \geqslant 0$, is totally positive and so is any ϕ_m, $m \geqslant 0$. □

4. Examples

This section is devoted to some examples showing the performances of the constructed nonstationary subdivision schemes. We consider, in particular, the case when $n = 2, 3, 4$, corresponding to C^1, C^2 and C^3 limit functions, respectively; moreover, we fix $\mu = 4$. By substituting these values in the explicit expression of the masks (3.5) and (3.6), we obtain the numerical values listed in Tables 1–3 for the first 11 values of k. For shortness the values in the tables are rounded to the seventh digit although the coefficients of the masks are explicit so that they can be evaluated with higher precision. We remark that the implementation of the proposed nonstationary method has the same cost of a stationary one once the table of the coefficients is constructed.

Next, the nonstationary subdivision algorithm corresponding to the masks given above, are used to generate both the limit functions ϕ_0 and the limit curves relative to a particular control polygon.

Table 1
Coefficients of the nonstationary masks $\boldsymbol{a}^k$, $k = 0, \ldots, 10$, for $n = 2$ and $\mu = 4$

k	a_0^k	a_1^k	a_2^k	a_3^k
0	0	1	1	0
1	0.125	0.875	0.875	0.125
2	0.2394008	0.7605992	0.7605992	0.2394008
3	0.2478698	0.7521302	0.7521302	0.2478698
4	0.2493240	0.7506760	0.7506760	0.2493240
5	0.2497229	0.7502771	0.7502771	0.2497229
6	0.2498663	0.7501337	0.7501337	0.2498663
7	0.2499278	0.7500722	0.7500722	0.2499278
8	0.2499577	0.7500423	0.7500423	0.2499577
9	0.2499736	0.7500264	0.7500264	0.2499736
10	0.2499827	0.7500173	0.7500173	0.2499827

The numerical values are rounded to the seventh digit.

Table 2
Coefficients of the nonstationary masks $\boldsymbol{a}^k$, $k=0,\ldots,10$, for $n=3$ and $\mu=4$

k	a_0^k	a_1^k	a_2^k	a_3^k	a_4^k
0	0	0.5	1	0.5	0
1	0.0625	0.5	0.875	0.5	0.0625
2	0.1197004	0.5	0.7605992	0.5	0.1197004
3	0.1239349	0.5	0.7521302	0.5	0.1239349
4	0.1246620	0.5	0.7506760	0.5	0.1246620
5	0.1248614	0.5	0.7502771	0.5	0.1248614
6	0.1249332	0.5	0.7501337	0.5	0.1249332
7	0.1249639	0.5	0.7500722	0.5	0.1249639
8	0.1249788	0.5	0.7500423	0.5	0.1249788
9	0.1249868	0.5	0.7500264	0.5	0.1249868
10	0.1249913	0.5	0.7500173	0.5	0.1249913

The numerical values are rounded to the seventh digit.

Table 3
Coefficients of the nonstationary masks $\boldsymbol{a}^k$, $k=0,\ldots,10$, for $n=4$ and $\mu=4$

k	a_0^k	a_1^k	a_2^k	a_3^k	a_4^k	a_5^k
0	0	0.25	0.75	0.75	0.25	0
1	0.03125	0.28125	0.6875	0.6875	0.28125	0.03125
2	0.0598502	0.3098502	0.6302996	0.6302996	0.3098502	0.0598502
3	0.0619674	0.3119674	0.6260651	0.6260651	0.3119674	0.0619674
4	0.0623310	0.3123310	0.6253380	0.6253380	0.3123310	0.0623310
5	0.0624307	0.3124307	0.6251385	0.6251385	0.3124307	0.0624307
6	0.0624666	0.3124666	0.6250668	0.6250668	0.3124666	0.0624666
7	0.0624820	0.3124820	0.6250361	0.6250361	0.3124820	0.0624820
8	0.0624894	0.3124894	0.6250211	0.6250211	0.3124894	0.0624894
9	0.0624934	0.3124934	0.6250132	0.6250132	0.3124934	0.0624934
10	0.0624957	0.3124957	0.6250087	0.6250087	0.3124957	0.0624957

The numerical values are rounded to the seventh digit.

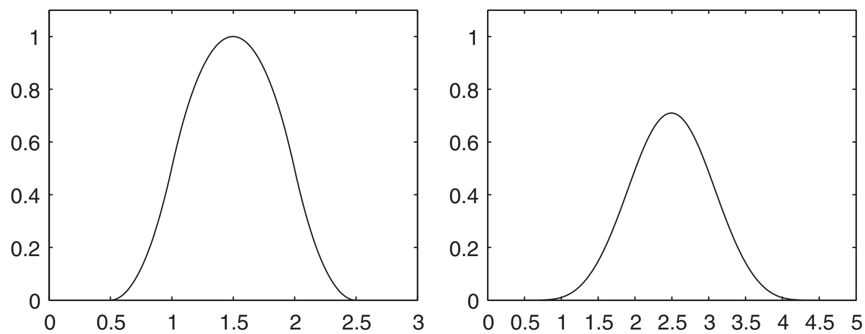

Fig. 2. Graphs of ϕ_0 for $n=2$ (left) and $n=4$ (right) when $\mu=4$.

We display in Fig. 2 the graphs of the limit function ϕ_0 obtained for $n=2$ and $n=4$. Observe that for $n=2$ ϕ_0 is C^1 and has support $[\frac{1}{2},\frac{5}{2}]$, while for $n=4$ ϕ_0 is C^3 and has support $[\frac{1}{2},\frac{9}{2}]$.

Just to give an idea of the behavior of the nonstationary subdivision scheme, in Fig. 3 the graphs of the starting control polygon and of the polygons obtained in the first 11 iterations are displayed for $n=4$ and $\mu=4$.

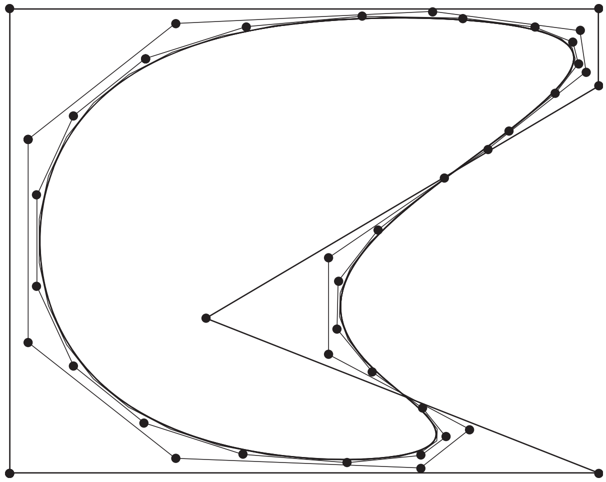

Fig. 3. The behavior of the subdivision algorithm for the first 11 iterations. The control points and the refined points of the first two iterations are marked with circles.

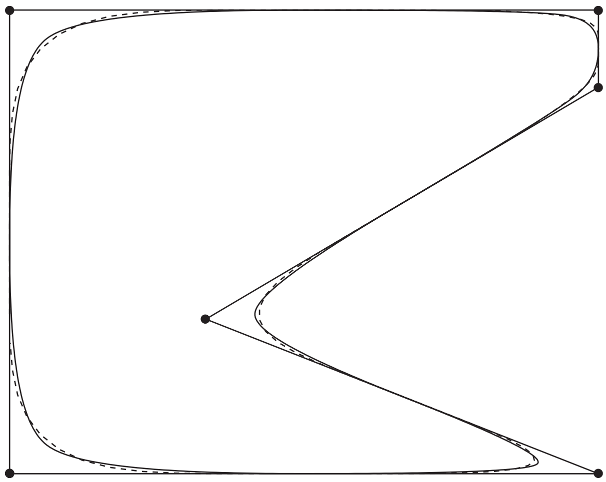

Fig. 4. The nonstationary curve for $n = 2$ and $\mu = 4$ (solid line) in comparison with the nonstationary up-function curve (dashed-line).

In Fig. 4 the limit curves obtained by the C^1 nonstationary subdivision scheme with $n = 2$ and $\mu = 4$ is compared with the curve obtained by the C^∞ nonstationary scheme

$$\boldsymbol{b}^k = \{b_\alpha^k\}_{\alpha=0}^{k+1}, \quad b_\alpha^k = \frac{1}{2^k}\binom{k+1}{\alpha}, \quad k = 0, 1, \ldots,$$

which gives rise to the up-function when applied to the delta-sequence.

Finally, in Figs. 5–7 the graphs of the limit curves obtained after 11 iterations of the nonstationary subdivision schemes for $n = 2, 3, 4$ and $\mu = 4$ are displayed (solid line). For each value of n, the limit curves obtained by the stationary subdivision schemes corresponding to the B-spline of degree n (dashed line) and $n + 1$ (dash-dotted line) are also displayed. We recall that, while for a given n the limit function ϕ_0 has support of length n and smoothness $n - 1$, the B-spline of degree n has support of length $n + 1$ and smoothness $n - 1$. The comparisons show that, for each value of n, the nonstationary curve is nearer the control polygon than both the two B-spline curves.

The examples above show that the proposed nonstationary schemes improve the behavior of the starting GP functions and, in particular, of the B-splines since, for a given n, they generate a limit function more regular than the B-spline

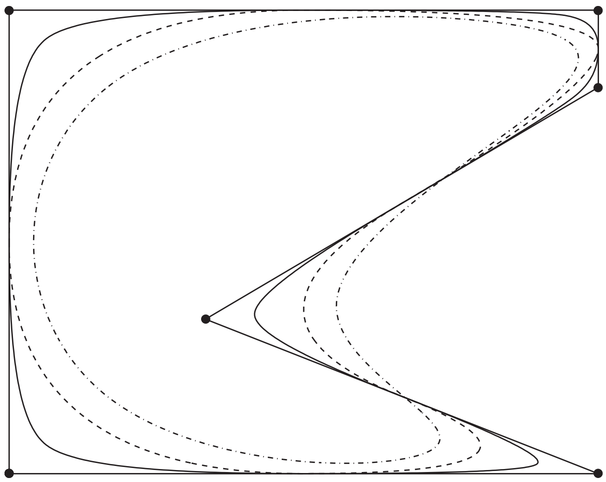

Fig. 5. The nonstationary curve for $n = 2$ and $\mu = 4$ (solid line) in comparison with the B-spline curves for $n = 2$ (dashed-line) and $n = 3$ (dash-dotted line).

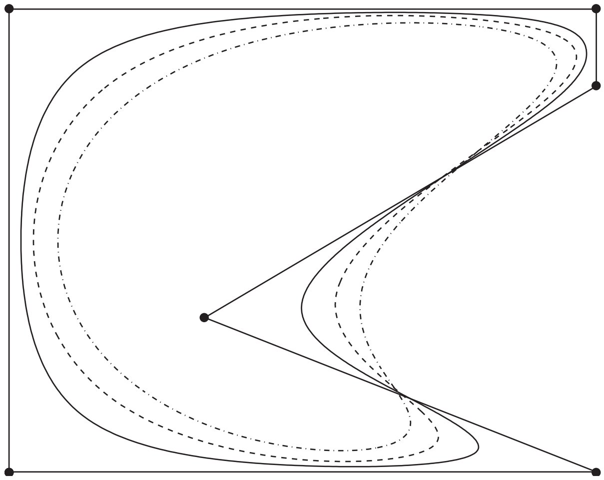

Fig. 6. The nonstationary curve for $n = 3$ and $\mu = 4$ (solid line) in comparison with the B-spline curves for $n = 3$ (dashed-line) and $n = 4$ (dash-dotted line).

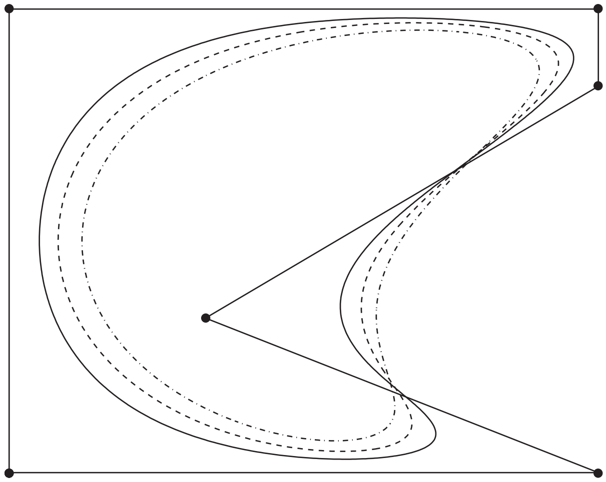

Fig. 7. The nonstationary curve for $n = 4$ and $\mu = 4$ (solid line) in comparison with the B-spline curves for $n = 4$ (dashed-line) and $n = 5$ (dash-dotted line).

of the same support, and at the same time more localized with respect to the B-spline having the same smoothness. Moreover, when $n=2$ the limit curve generated by the proposed nonstationary scheme, which is C^1, is similar to that one generated by the nonstationary up-function scheme, which is C^∞.

The results above encourage us to extend the our construction of subdivision schemes to the bivariate case. A family of stationary schemes depending on free parameters has been introduced in [2] by means of directional convolution. The construction of nonstationary schemes based on a procedure similar to the univariate one is at present under investigation.

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